$$E_k(z) = \frac{1}{2} \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ c,\,d \text{ coprime}}} \frac{1}{(cz+d)^k}$$

$k \geq 4$ even.

modular forms of wt k.

of level Γ_1.

$$G_k(z) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{1}{(mz+n)^k}$$

$$G_k(z) = \frac{1}{2} \sum_{r=1}^{\infty} \sum_{\substack{c,d\in\mathbb{Z} \\ c,\,d \text{ coprime}}} \frac{1}{r^k (cz+d)^k}$$

$$= \sum_{r=1}^{\infty} \frac{1}{r^k} \cdot E_k(z) = \underbrace{\zeta(k)}_{\text{Riemann zeta function.}} E_k(z).$$

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z)$$

known from "last time"

Prop: $M_*(\Gamma_1) = \mathbb{C}[E_4, E_6]$

Cor: For $k \geq 0$ even, $\dim M_k(\Gamma_1) \overset{\leq}{=} \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \bmod 12 \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \bmod 12. \end{cases}$

Pf: E_4, E_6 are alg. independent.

$E_4^3 \quad E_6^2$ not proportional

Otherwise $\lambda E_4^3 = E_6^2$.

Consider $f = \dfrac{E_6}{E_4}$ meromorphic modular form of wt 2

$f^2 = \lambda E_4 \Rightarrow f$ holomorphic

$f \in M_2(\Gamma_1)$. $\leftarrow \dim \leq 0$.

$\Rightarrow f = 0$ ↯

$\mathbb{C}[E_4, E_6] \subset M_*(\Gamma_1)$

wt k part

$\text{Span}\{E_4^a E_6^b \mid 4a + 6b = k\}$

$\dim$ wt k part of $\mathbb{C}[E_4, E_6] = \begin{cases} \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \bmod 12 \\ \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \bmod 12 \end{cases}$

$\Rightarrow$ forced to have equalities.

$\square$

Fourier expansion (q-expansion) $\qquad q = e^{2\pi i z}$.

Prop: $$G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n)\, q^n.$$

B_k $\quad$ k-th Bernoulli number $\qquad \dfrac{x}{e^x - 1} = \displaystyle\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$.

$B_k = 0$ if $k > 1$ odd.

$B_2 = \frac{1}{6}$, $\; B_4 = -\frac{1}{30}$, $\; \ldots$, $\; B_{12} = -\frac{691}{2730}$.

$$\sigma_k(n) = \sum_{d \mid n} d^k.$$

Pf: make use of Taylor expansion of $\dfrac{\pi}{\tan(\pi z)}$

$$\frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{\pi}{\tan \pi z} \right) = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^k} \longrightarrow \text{Eisenstein series}$$

$\square$

Applications: use $\dim M_k(\Gamma_1) = 1$ for some k.

$\rightsquigarrow$ get relations among E_k.

$\rightsquigarrow$ get relations on Fourier coeff. $\sigma_k(n)$.

$\dim M_8(\Gamma_1) = 1$.

$$E_4^2 = E_8 \Rightarrow \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n) - \sigma_3(n)}{120}$$

$$\begin{array}{ccc} E_4^2 & & E_8 \\ \downarrow & & \downarrow \\ \sigma_3 & & \sigma_7 \end{array}$$

$\underline{E_2}$

$$E_k(z) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 \\ c,\, d \text{ coprime}}} \frac{1}{(cz+d)^k}$$ not abs. conv. when $k=2$.

$$G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$ is abs. conv. when $k=2$.

so we just define $G_2(z)$ by this series.

$$G_2(z) = \frac{1}{2} \sum_m \sum_n \frac{1}{(mz+n)^k}$$

You have to take summation in this particular order.

G_2 is holomorphic but not modular quasimodular.

G_2

E_2

$G_2^*(z) = G_2(z) + \frac{1}{8\pi y}$ is not holomorphic but modular.

almost holomorphic modular form,

<u>Disriminant function Δ</u>

$$\Delta(z) = q \prod_{i=1}^{\infty} (1-q^n)^{24}$$

$$\frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) = E_2(z)$$

Make use of transformation property of E_2

$\Rightarrow \Delta(z) \in M_{12}(\Gamma_1)$ $\dim M_{12}(\Gamma_1) = 2$.

spanned by E_4^3, E_6^2

$$\Delta(z) = \frac{1}{1728}\left(E_4^3(z) - E_6^2(z)\right)$$

$j(z) := \frac{E_4^3(z)}{\Delta(z)}$ j-invariant function modular of wt 0

exp growth near ∞.

$$j : \Gamma_1 \backslash \mathfrak{h} \xrightarrow{\sim} \mathbb{C}$$

$$\overline{\Gamma_1 \backslash \mathfrak{h}} \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$$

$$\Delta(z) = \sum_{n=1}^{\infty} \tau(n) q^n$$

cuspidal cusp forms.

Consequence of Weil conjecture proved by Deligne

Ramanujan's conjecture

$$|\tau(p)| \leq 2 p^{\frac{12-1}{2}}$$

Generalised Ramanujan's Conjecture is wide open.

Ramanujan's congruence

$$G_{12}(z) = \Delta(z) + \frac{691}{156} \cdot \left(\frac{E_4(z)^3}{720} + \frac{E_6(z)^2}{1008} \right)$$

$$\sigma_{11}(n) \equiv \tau(n) \pmod{691}$$